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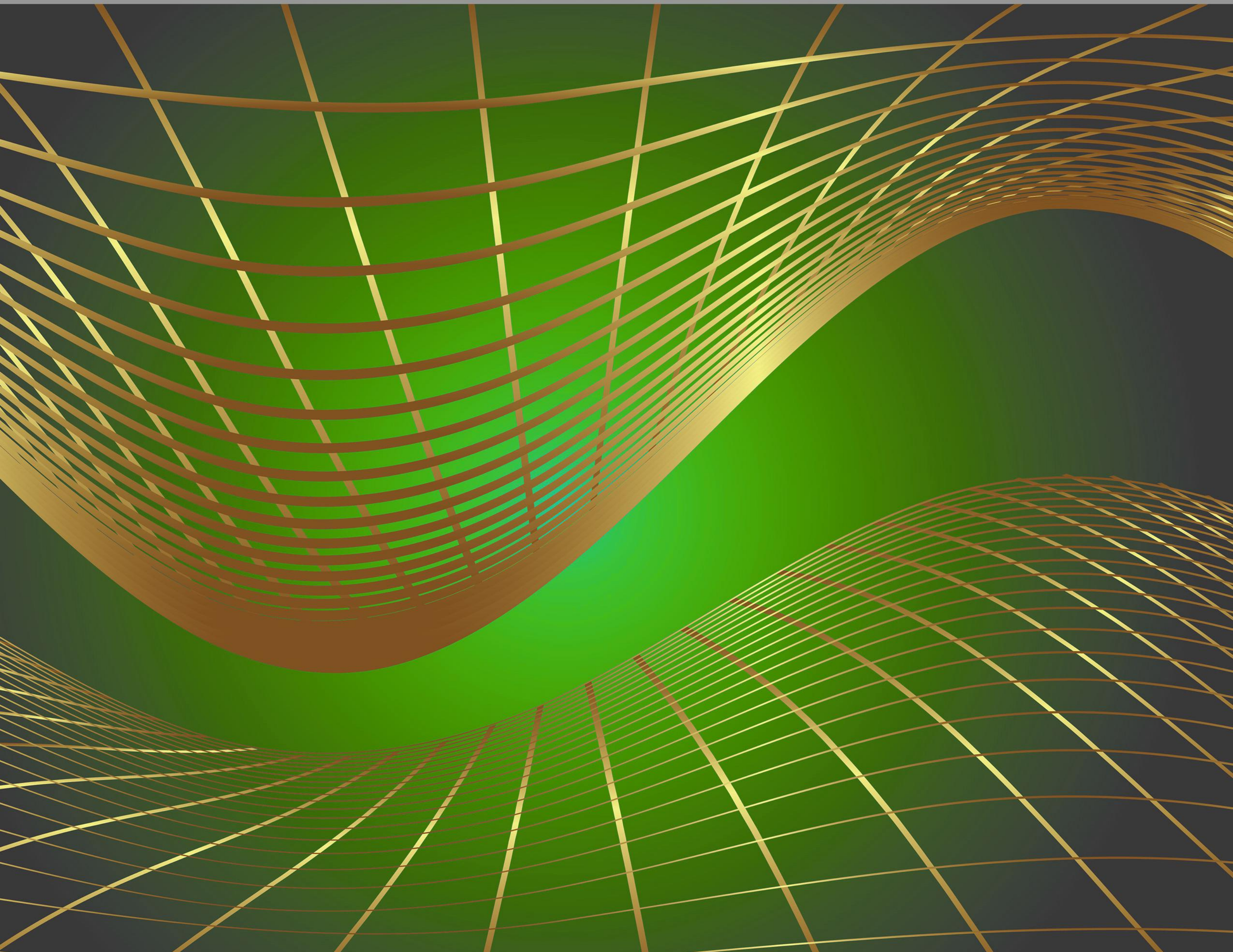

Math 4 Week of April 6th Enrichment

CK-12
Mark Spong
Kaitlyn Spong

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CHAPTER 1 Finance

Chapter Outline

Here you will review concepts of exponential growth and geometric series with a focus on the relationship between time and money.

1.1 Simple Interest

Learning Objectives

Here you'll learn to calculate the effect of time on the balance of a savings account growing by simple interest.

The basic concept of **interest** is that a dollar today will be worth more than a dollar next year. If a person deposits \$100 into a bank account today at 6% simple interest, then in one year the bank owes the person that \$100 plus a few dollars more. If the person decides to leave it in the account and keep earning the interest, then after two years the bank would owe the person even more money. How much interest will the person earn each year? How much money will the person have after two years?

Simple Interest

Simple interest is defined as interest that only accumulates on the initial money deposited in the account. This initial money is called the **principal**. In the real world, most companies do not use simple interest because it is considered too simple and instead use compound interest which compounds on itself. You will practice with simple interest here because it introduces the concept of the time value of money and that a dollar saved today is worth slightly more than a dollar in one year.

The formula for simple interest has 4 variables and all the problems and examples will give 3 and your job will be to find the unknown quantity using rules of Algebra.

FV means **future value** and it stands for the amount in the account at some future time t.

PV means **present value** and it stands for the amount in the account at time 0.

t means **time** (usually years) that has elapsed between the present value and the future value. The value of t indicates how long the money has been accumulating interest.

i means the **simple interest rate**. If the interest rate is 6%, in the formula you will use the decimal version of 0.06. Here is the formula that shows the relationship between FV and PV.

$FV = PV(1+t\cdot i)$

Let's say Linda invested \$1,000 for her child's college education and she saved it for 18 years at a bank which offered 5% simple interest. To find out how much she has at the end of 18 years, first identify known and unknown quantities.

$$\begin{aligned}PV &= \$1,000\\ t &= 18 \textit{ years}\\ i &= 0.05\\ FV &= \text{unknown so you will use } x\end{aligned}$$

Then substitute the values into the formula and solve to find the future value.

$$
\begin{aligned}
FV &= PV(1+t\cdot i)\\
x &= 1,000(1+18\cdot 0.05)\\
x &= 1,000(1+0.90)\\
x &= 1,000(1.9)\\
x &= 1,900
\end{aligned}
$$

Linda initially had $1,000, but 18 years later with the effect of 5% simple interest, that money grew to $1,900.

Examples

Example 1

Earlier, you were asked about the how much a person who deposits $100 today at 6% simple interest will have in one year and in two years. That person will have have $106 in one year and $112 in two years.

Example 2

Tory put $200 into a bank account that earns 8% simple interest. How much interest does Tory earn each year and how much does she have at the end of 4 years?

First you will focus on the first year and identify known and unknown quantities.

$$
\begin{aligned}
PV &= \$200\\
t &= 1\ year\\
i &= 0.08\\
FV &= \text{unknown so we will use } x
\end{aligned}
$$

Second, you will substitute the values into the formula and solve to find the future value.

The third thing you need to do is interpret and organize the information. Tory had $200 to start with and then at the end of one year she had $216. The additional $16 is interest she has earned that year. Since the account is simple interest, she will keep earning $16 dollars every year because her principal remains at $200. The $16 of interest earned that first year just sits there earning no interest of its own for the following three years.

TABLE 1.1:

Year	Principal at Beginning of Year	Interest Earned that Year	Total Interest Earned
1	200	$200 \times .08 = 16$	16
2	200	16	32
3	200	16	48
4	200	16	64

At the end of 4 years, Tory will have \$264 on her account. \$64 will be interest. She earned \$16 in interest each year.

Example 3

Amy has \$5000 to save and she wants to buy a car for \$10,000. For how many years will she need to save if she earns 10% simple interest? On the other hand, what will the simple interest rate need to be if she wants to save enough money in 15 years?

Notice that there are two separate problems. Let's start with the first problem and identify known and unknown quantities.

$$\begin{aligned} PV &= 5,000 \\ FV &= 10,000 \\ i &= 0.10 \\ t &= ? \end{aligned}$$

Now substitute and solve for t.

Now let's focus on the second problem and go through the process of identifying known and unknown quantities, substituting and solving.

To answer the first question, Amy would need to save for 10 years getting a simple interest rate of 10%. For the second question, she would need to save for 15 years at a simple interest rate of about 6.667%.

Example 4

How long will it take \$3,000 to grow to \$4,000 at 4% simple interest?

$PV = 3,000,\ t = ?,\ i = 0.04,\ FV = 4,000$

Example 5

What starting balance grows to $5,000 in 5 years with 10% simple interest?

$PV = ?,\ FV = 5,000,\ t = 5,\ i = 0.10$

Review

1. How much will a person have at the end of 8 years if they invest $3,000 at 4.5% simple interest?
2. How much will a person have at the end of 6 years if they invest $2,000 at 3.75% simple interest?
3. How much will a person have at the end of 12 years if they invest $1,500 at 7% simple interest?
4. How much interest will a person earn if they invest $10,000 for 10 years at 5% simple interest?
5. How much interest will a person earn if they invest $2,300 for 49 years at 3% simple interest?
6. How long will it take $2,000 to grow to $5,000 at 3% simple interest?
7. What starting balance grows to $12,000 in 8 years with 10% simple interest?
8. Suppose you have $3,000 and want to have $35,000 in 25 years. What simple interest rate will you need?
9. How long will it take $1,000 to grow to $4,000 at 8% simple interest?
10. What starting balance grows to $9,500 in 4 years with 6.5% simple interest?
11. Suppose you have $1,500 and want to have $8,000 in 15 years. What simple interest rate will you need?
12. Suppose you have $800 and want to have $6,000 in 45 years. What simple interest rate will you need?
13. What starting balance grows to $2,500 in 2 years with 1.5% simple interest?
14. Suppose you invest $4,000 which earns 5% simple interest for the first 12 years and then 8% simple interest for the next 8 years. How much money will you have after 20 years?
15. Suppose you invest $10,000 which earns 2% simple interest for the first 8 years and then 5% simple interest for the next 7 years. How much money will you have after 15 years?

Review (Answers)

To see the Review answers, open this PDF file and look for section 13.1.

Principal is the amount initially deposited into the account. *Notice the spelling is principal, not principle*.

Interest is the conversion of time into money.

1.2 Compound Interest per Year

Learning Objectives

Here you'll explore how to compute an investment's growth given time and a compound interest rate.

If a person invests \$100 in a bank with 6% simple interest, they earn \$6 in the first year and \$6 again in the second year totaling \$112. If this was really how interest operated with most banks, then someone clever may think to withdraw the \$106 after the first year and immediately reinvest it. That way they earn 6% on \$106. At the end of the second year, the clever person would have earned \$6 like normal, plus an extra .36 cents totaling \$112.36. Thirty six cents may seem like not very much, but how much more would a person earn if they saved their \$100 for 50 years at 6% compound interest rather than at just 6% simple interest?

Compound Interest Per Year

Compound interest allows interest to grow on interest. As with simple interest, *PV* is defined as present value, *FV* is defined as future value, *i* is the interest rate, and *t* is time. The formulas for simple and compound interest look similar, so be careful when reading problems in determining whether the interest rate is simple or compound. The following table shows the amount of money in an account earning compound interest over time:

TABLE 1.2:

Year	Amount Ending in Account
1	$FV = PV(1+i)$
2	$FV = PV(1+i)^2$
3	$FV = PV(1+i)^3$
4	$FV = PV(1+i)^4$
...	
t	$FV = PV(1+i)^t$

For now you should assume that you are compounding the interest once a year or annually. An account with a present value of *PV* that earns compound interest at *i* percent annually for *t* years has a future value of *FV* shown below:

$$FV = PV(1+i)^t$$

Applying this formula for years 1, 2, 3, and 4 for an initial deposit of \$100 at 3% compound interest, you would get the following results:

$PV = 100,\ i = 0.03,\ t = 1,2,3 \textit{ and } 4,\ FV = ?$

TABLE 1.3:

Year	Amount ending in Account
1	$FV = 100(1+0.03) = 103.00$

TABLE 1.3: (continued)

2	$FV = 100(1+0.03)^2 = 106.09$
3	$FV = 100(1+0.03)^3 \approx 109.27$
4	$FV = 100(1+0.03)^4 \approx 112.55$

Calculator shortcut: When doing repeated calculations that are just 1.03 times the result of the previous calculation, use the <ANS> button to create an entry that looks like <Ans*1.03>. Then, pressing enter repeatedly will rerun the previous entry producing the values on the right.

Examples

Example 1

Earlier you were introduced to a concept problem contrasting \$100 for 50 years at 6% compound interest versus 6% simple. Now you can calculate how much more powerful compound interest is.

$PV = 100,\ t = 50,\ i = 6\%,\ FV = ?$

Simple interest:

$FV = PV(1+t \cdot i) = 100(1+50 \cdot 0.06) = 400$

Compound interest:

$FV = PV(1+i)^t = 100(1+0.06)^{50} \approx 1,842.02$

It is remarkable that simple interest grows the balance of the account to \$400 while compound interest grows it to about \$1,842.02. The additional money comes from interest growing on interest repeatedly.

Example 2

How much will Kyle have in a savings account if he saves \$3,000 at 4% compound interest for 10 years?

$PV = 3,000,\ i = 0.04,\ t = 10\ years,\ FV = ?$

$$FV = PV(1+i)^t$$
$$FV = 3000(1+0.04)^{10} \approx \$4,440.73$$

Example 3

How long will it take money to double if it is in an account earning 8% compound interest?

There are two ways you can solve this problem, through estimation or through computation.

Estimation Solution: The rule of 72 is an informal means of estimating how long it takes money to double. It is useful because it is a calculation that can be done mentally that can yield surprisingly accurate results. This can be very helpful when doing complex problems to check and see if answers are reasonable. The **rule of 72** simply states $\frac{72}{i} \approx t$ where i is written as an integer (i.e. 8% would just be 8).

In this case $\frac{72}{8} = 9 \approx t$, so it will take about 9 years.

Exact Solution: Since there is no initial value you are just looking for some amount to double. You can choose any amount for the present value and double it to get the future value even though specific numbers are not stated in the problem. Here you should choose 100 for PV and 200 for FV.

$PV = 100,\ FV = 200,\ i = 0.08,\ t = ?$

$$\begin{aligned}
FV &= PV(1+i)^t \\
200 &= 100(1+0.08)^t \\
2 &= 1.08^t \\
\ln 2 &= \ln 1.08^t \\
\ln 2 &= t \cdot \ln 1.08 \\
t &= \frac{\ln 2}{\ln 1.08} = 9.00646
\end{aligned}$$

It will take just over 9 years for money (any amount) to double at 8%. This is extraordinarily close to your estimation and demonstrates how powerful the Rule of 72 can be in estimation.

Example 4

How long will it take money to double at 6% compound interest? Estimate using the rule of 72 and also find the exact answer.

Estimate: $\frac{72}{6} = 12 \approx$ years it will take to double

$PV = 100,\ FV = 200,\ i = 0.06,\ t = ?$

$$\begin{aligned}
200 &= 100(1+0.06)^t \\
2 &= (1.06)^t \\
\ln 2 &= \ln 1.06^t = t\ \ln 1.06 \\
t &= \frac{\ln 2}{\ln 1.06} \approx 11.89\ years
\end{aligned}$$

Example 5

What compound interest rate is needed to grow \$100 to \$120 in three years?

$PV = 100,\ FV = 120,\ t = 3,\ i = ?$

$$\begin{aligned}
FV &= PV(1+i)^t \\
120 &= 100(1+i)^3 \\
[1.2]^{\frac{1}{3}} &= [(1+i)^3]^{\frac{1}{3}} \\
[1.2]^{\frac{1}{3}} &= 1+i \\
i &= 1.2^{\frac{1}{3}} - 1 \approx 0.06266
\end{aligned}$$

Review

For problems 1-10, find the missing value in each row using the compound interest formula.

TABLE 1.4:

Problem Number	*PV*	*FV*	*t*	*i*
1.	\$1,000		7	1.5%
2.	\$1,575	\$2,250	5	
3.	\$4,500	\$5,534.43		3%
4.		\$10,000	12	2%
5.	\$1,670	\$3,490	10	
6.	\$17,000	\$40,000	25	
7.	\$10,000	\$17,958.56		5%
8.		\$50,000	30	8%
9.		\$1,000,000	40	6%
10.	\$10,000		50	7%

11. How long will it take money to double at 4% compound interest? Estimate using the rule of 72 and also find the exact answer.

12. How long will it take money to double at 3% compound interest? Estimate using the rule of 72 and also find the exact answer.

13. Suppose you have \$5,000 to invest for 10 years. How much money would you have in 10 years if you earned 4% simple interest? How much money would you have in 10 years if you earned 4% compound interest?

14. Suppose you invest \$4,000 which earns 5% compound interest for the first 12 years and then 8% compound interest for the next 8 years. How much money will you have after 20 years?

15. Suppose you invest \$10,000 which earns 2% compound interest for the first 8 years and then 5% compound interest for the next 7 years. How much money will you have after 15 years?

Review (Answers)

To see the Review answers, open this PDF file and look for section 13.2.

1.3 Compound Interest per Period

Learning Objectives

Here you'll learn to compute future values with interest that accumulates semi-annually, monthly, daily, etc.

Clever Carol went to her bank which was offering 12% interest on its savings account. She asked very nicely if instead of having 12% at the end of the year, if she could have 6% after the first 6 months and then another 6% at the end of the year. Carol and the bank talked it over and they realized that while the account would still seem like it was getting 12%, Carol would actually be earning a higher percentage. How much more will Carol earn this way?

Compound Interest Per Period of Time

Consider a bank that compounds and adds interest to accounts k times per year. If the original percent offered is 12% then in one year that interest can be compounded:

- Once, with 12% at the end of the year $(k = 1)$
- Twice (semi-annually), with 6% after the first 6 months and 6% after the last six months $(k = 2)$
- Four times (quarterly), with 3% at the end of each 3 months $(k = 4)$
- Twelve times (monthly), with 1% at the end of each month $(k = 12)$

The intervals could even be days, hours or minutes. This is called the length of the **compounding period.** The number of compounding periods is how often interested is compounded. When intervals become small so does the amount of interest earned in that period, but since the intervals are small there are more of them. This effect means that there is a much greater opportunity for interest to compound.

Nominal interest is a number that resembles an interest rate, but it really is a sum of compound interest rates. A nominal rate of 12% compounded monthly is really 1% compounded 12 times. The formula for interest compounding k times per year for t years at a nominal interest rate i with present value PV and future value FV is:

$FV = PV\left(1+\frac{i}{k}\right)^{kt}$

As with simple interest and compound interest, the nominal rate of interest is represented with the letter i in this formula, but the resulting rate is computed differently. A nominal rate of 12% may actually yield more than 12%.

Let's apply the formula above to an investment of \$300 at a rate of 12% compounded monthly. If you wanted to know the amount of money the person would have after 4 years, you would take the following steps:

$FV = ?,\ PV = 300,\ t = 4,\ k = 12,\ i = 0.12$

$FV = PV\left(1+\frac{i}{k}\right)^{kt} = 300\left(1+\frac{0.12}{12}\right)^{12\cdot 4} \approx 483.67$

Note: A very common mistake when typing the values into a calculator is using an exponent of 12 and then multiplying the whole quantity by 4 instead of using an exponent of $(12\cdot 4) = 48$.

Examples

Example 1

Earlier, you were asked about Clever Carol and the difference in amount of money she would have if her interest was compounded once a year versus twice a year. If Clever Carol earned the 12% at the end of the year she would earn \$12 in interest in the first year. If she compounds it $k = 2$ times per year then she will end up earning:

$FV = PV\left(1+\frac{i}{k}\right)^{kt} = 100\left(1+\frac{.12}{2}\right)^{2\cdot 1} = \$112.36$

Example 2

How many years will Matt need to invest his money at 6% compounded daily $(k = 365)$ if he wants his \$3,000 to grow to \$5,000?

$FV = 5,000,\ PV = 3,000,\ k = 365,\ i = 0.06,\ t = ?$

$$
\begin{aligned}
FV &= PV\left(1+\frac{i}{k}\right)^{kt}\\
5,000 &= 3,000\left(1+\frac{0.06}{365}\right)^{365t}\\
\frac{5}{3} &= \left(1+\frac{0.06}{365}\right)^{365t}\\
\ln\frac{5}{3} &= \ln\left(1+\frac{0.06}{365}\right)^{365t}\\
\ln\frac{5}{3} &= 365t\cdot\ln\left(1+\frac{0.06}{365}\right)\\
t &= \frac{\ln\frac{5}{3}}{365\cdot\ln\left(1+\frac{0.06}{365}\right)} = 8.514\ years
\end{aligned}
$$

Example 3

What nominal interest rate compounded quarterly doubles money in 5 years?

$FV = 200,\ PV = 100,\ k = 4,\ i = ?,\ t = 5$

Example 4

How much will Steve have in 8 years if he invests \$500 in a bank that offers 8% compounded quarterly?

$PV = 500,\ t = 8,\ i = 8\%,\ FV = ?,\ k = 4$

$FV = PV\left(1+\frac{i}{k}\right)^{kt} = 500\left(1+\frac{0.08}{4}\right)^{4\cdot 8} = \$942.27$

Example 5

How many years will Mark need to invest his money at 3% compounded weekly $(k = 52)$ if he wants his \$100 to grow to \$400?

$FV = 400,\ PV = 100,\ k = 52,\ i = 0.03,\ t = ?$

$$\begin{aligned}
FV &= PV\left(1+\frac{i}{k}\right)^{kt}\\
400 &= 100\left(1+\frac{0.03}{52}\right)^{52\cdot t}\\
\ln 4 &= \ln\left(1+\frac{0.03}{52}\right)^{52t}\\
t &= \frac{\ln 4}{52\cdot\ln\left(1+\frac{0.03}{52}\right)} = 46.22\ years
\end{aligned}$$

Review

1. What is the length of a compounding period if $k = 12$?
2. What is the length of a compounding period if $k = 365$?
3. What would the value of k be if interest was compounded every hour?
4. What would the value of k be if interest was compounded every minute?
5. What would the value of k be if interest was compounded every second?

For problems 6-15, find the missing value in each row using the compound interest formula.

TABLE 1.5:

Problem Number	PV	FV	t	i	k
6.	$1,000		7	1.5%	12
7.	$1,575	$2,250	5		2
8.	$4,000	$5,375.67		3%	1
9.		$10,000	12	2%	365
10.	$10,000		50	7%	52
11.	$1,670	$3,490	10		4
12.	$17,000	$40,000	25		12
13.	$12,000		3	5%	365
14.		$50,000	30	8%	4
15.		$1,000,000	40	6%	2

Review (Answers)

To see the Review answers, open this PDF file and look for section 13.3.

1.4 Continuous Interest

Learning Objectives

Here you'll learn to use the force of interest to compute future values when interest is being compounded continuously.

Clever Carol realized that she makes more money when she convinces the bank to give her 12% in two chunks of 6% than only one time at 12%. Carol knew she could convince them to give her 1% at the end of each month for a total of 12% which would be even more than the two chunks of 6%. As Carol makes the intervals smaller and smaller, does she earn more and more money from the bank? Does this extra amount ever stop or does it keep growing forever?

Continuous Interest

Calculus deals with adding up an infinite number of infinitely small amounts. Using calculus, we can derive the value e to help us understand what happens as k, the number of compounding periods, approaches infinity. The number e is used frequently in finance and other fields to represent this type of continuous growth.

$e \approx \left(1+\frac{1}{k}\right)^k \approx 2.71828\ldots$ as k approaches infinity

This means that even when there are an infinite number of infinitely small compounding periods, there will be a limit on the interest earned in a year. The term for infinitely small compounding periods is continuous compounding. A **continuously compounding interest rate** is the rate of growth proportional to the amount of money in the account at every instantaneous moment in time. It is equivalent to infinitely many but infinitely small compounding periods.

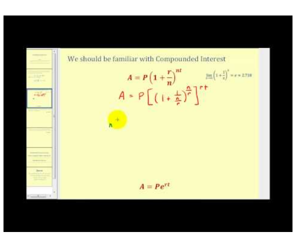
The formula for finding the future value of a present value invested at a continuously compounding interest rate r for t years is:

$FV = PV \cdot e^{rt}$

Applying this formula, you can determine what the future value of \$360 invested for 6 years at a continuously compounding rate of 5% is.

$FV = ?,\ PV = 360,\ r = 0.05,\ t = 6$

$FV = PV \cdot e^{rt} = 360e^{0.05 \cdot 6} = 360e^{0.30} \approx 485.95$



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Examples

Example 1

Earlier, you were asked to compare the amount of money Clever Carol would make using different rates of compounding. Clever Carol could calculate the returns on each of the possible compounding periods for one year.

For once per year, $k = 1$:

$FV = PV(1+i)^t = 100(1+0.12)^1 = 112$

For twice per year, $k = 2$:

$FV = PV(1+i)^t = 100\left(1+\frac{0.12}{2}\right)^2 = 112.36$

For twelve times per year, $k = 12$:

$FV = PV(1+i)^t = 100\left(1+\frac{0.12}{12}\right)^{12} \approx 112.68$

At this point Carol might notice that while she more than doubled the number of compounding periods, she did not more than double the extra pennies. The growth slows down and approaches the continuously compounded growth result.

For continuously compounding interest:

$FV = PV \cdot e^{rt} = 100 \cdot e^{0.12 \cdot 1} \approx 112.75$

No matter how many times Clever Carol might convince her bank to compound the 12% over the course of each year, the most she can earn from the original \$100 is around \$12.75 in interest.

Example 2

What is the continuously compounding rate that will grow \$100 into \$250 in just 2 years?

$PV = 100,\ FV = 250,\ r = ?,\ t = 2$

$$\begin{aligned}
FV &= PV \cdot e^{rt} \\
250 &= 100 \cdot e^{2r} \\
2.5 &= e^{2r} \\
\ln 2.5 &= 2r \\
r &= \frac{\ln 2.5}{2} \approx 0.4581 = 45.81\%
\end{aligned}$$

Example 3

What amount invested at 7% continuously compounding yields \$1,500 after 8 years?

$PV = ?\ FV = 1,500,\ t = 8,\ r = 0.07$

$$\begin{aligned}
FV &= PV \cdot e^{rt} \\
1,500 &= PV \cdot e^{0.07 \cdot 8} \\
PV &= \frac{1,500}{e^{0.07 \cdot 8}} \approx \$856.81
\end{aligned}$$

Example 4

What is the future value of \$500 invested for 8 years at a continuously compounding rate of 9%?

$FV = 500e^{8 \cdot 0.09} \approx 1027.22$

Example 5

What is the continuously compounding rate which grows \$27 into \$99 in just 4 years?

$99 = 27e^{4r}$

Solving for r yields: $r = 0.3248 = 32.48\%$

Review

For problems 1-10, find the missing value in each row using the continuously compounding interest formula.

TABLE 1.6:

Problem Number	PV	FV	t	r
1.	\$1,000		7	1.5%
2.	\$1,575	\$2,250	5	
3.	\$4,500	\$5,500		3%
4.		\$10,000	12	2%
5.	\$1,670	\$3,490	10	
6.	\$17,000	\$40,000	25	
7.	\$10,000	\$18,000		5%
8.		\$50,000	30	8%
9.		\$1,000,000	40	6%
10.	\$10,000		50	7%

11. How long will it take money to double at 4% continuously compounding interest?

12. How long will it take money to double at 3% continuously compounding interest?

13. Suppose you have \$6,000 to invest for 12 years. How much money would you have in 10 years if you earned 3% simple interest? How much money would you have in 10 years if you earned 3% continuously compounding interest?

14. Suppose you invest \$2,000 which earns 5% continuously compounding interest for the first 12 years and then 8% continuously compounding interest for the next 8 years. How much money will you have after 20 years?

15. Suppose you invest \$7,000 which earns 1.5% continuously compounding interest for the first 8 years and then 6% continuously compounding interest for the next 7 years. How much money will you have after 15 years?

Review (Answers)

To see the Review answers, open this PDF file and look for section 13.4.

1.5 APR and APY (Nominal and Effective Rates)

Learning Objectives

Here you'll learn how to compare rates for loans and savings accounts to find more favorable deals.

In looking at an advertisement for a car you might see 2.5% APR financing on a $20,000 car. What does APR mean? What rate are they really charging you for the loan? Different banks may offer 8.1% annually, 8% compounded monthly or 7.9% compounded continuously. How much would you really be making if you put $100 in each bank? Which bank has the best deal?

Nominal and Effective Rates of Interest

A **nominal interest rate** is an interest rate in name only since a method of compounding needs to be associated with it in order to get a true effective interest rate. APR rates are nominal. **APR** stands for **Annual Percentage Rate**. The compounding periods are usually monthly, so typically $k = 12$.

An **annual effective interest rate** is the true interest that is being charged or earned. APY rates are effective rates. **APY** stands for **Annual Percentage Yield**. It is a true rate that states exactly how much money will be earned as interest.

Banks, car dealerships and all companies will often advertise the interest rate that is most appealing to consumers who don't know the difference between APR and APY. In places like loans where the interest rate is working against you, they advertise a nominal rate that is lower than the effective rate. On the other hand, banks want to advertise the highest rates possible on their savings accounts so that people believe they are earning more interest.

In order to calculate what you are truly being charged, or how much money an account is truly making, it is necessary to use what you have learned about compounding interest and continuous interest. Then, you can make an informed decision about what is best.

Take a credit card that advertises 19.9% APR (annual rate compounded monthly). Say you left $1000 unpaid, how much would you owe in a year?

First recognize that 19.9% APR is a nominal rate compounded monthly.

$FV = ?\ PV = 1000,\ i = .199,\ k = 12,\ t = 1$

$$FV = 1000\left(1 + \frac{0.199}{12}\right)^{12} \approx \$1,218.19$$

Notice that $1,218.19 is an increase of about 21.82% on the original $1,000. Many consumers expect to pay only $199 in interest because they misunderstood the term APR. The effective interest on this account is about 21.82%, which is more than advertised.

Another interesting note is that just like there are rounding conventions in this math text (4 significant digits or dollars and cents), there are legal conventions for rounding interest rate decimals. Many companies include an additional 0.0049% because it rounds down for advertising purposes, but adds additional cost when it is time to pay up. For the purposes of this concept, ignore this addition.

Examples

Example 1

Earlier, you were asked about financing a car and the difference between APR and APY. A loan that offers 2.5% APR that compounds monthly is really charging lightly more than 2.5% of the initial loan per year.

$\left(1+\frac{0.025}{12}\right)^{12} \approx 1.025288$

They are really charging about 2.529%.

The table below shows the APY calculations for three different banks offering 8.1% annually, 8% compounded monthly and 7.9% compounded continuously.

TABLE 1.7:

Bank A	Bank B	Bank C
$FV = PV(1+i)^t$ $FV = 100(1+0.081)$ $FV = \$108.1$ $APY = 8.1\ \%$	$FV = PV\left(1+\frac{i}{k}\right)^{kt}$ $FV = 100\left(1+\frac{0.08}{12}\right)^{12}$ $FV \approx 108.299$ $APY \approx 8.299\%$	$FV = PV \cdot e^{rt}$ $FV = 100e^{0.079}$ $FV\& \approx 108.22$ $APY \approx 8.22\%$

Even though Bank B does not seem to offer the best interest rate, or the most advantageous compounding strategy, it still offers the highest yield to the consumer.

Example 2

Three banks offer three slightly different savings accounts. Calculate the Annual Percentage Yield for each bank and choose which bank would be best to invest in.

Bank A offers 7.1% annual interest.

Bank B offers 7.0% annual interest compounded monthly.

Bank C offers 6.98% annual interest compounded continuously.

Since no initial amount is given, choose a PV that is easy to work with like \$1 or \$100 and test just one year so $t = 1$. Once you have the future value for 1 year, you can look at the percentage increase from the present value to determine the APY.

TABLE 1.8:

Bank A	Bank B	Bank C

TABLE 1.8: (continued)

$FV = PV(1+i)^t$ $FV = 100(1+0.071)$ $FV = \$107.1$ $APY = 7.1\%$	$FV = PV\left(1+\frac{i}{k}\right)^{kt}$ $FV = 100\left(1+\frac{0.07}{12}\right)^{12}$ $FV \approx 107.229$ $APY \approx 7.2290\%$	$FV = PV \cdot e^{rt}$ $FV = 100e^{.0698}$ $FV \approx 107.2294$ $APY \approx 7.2294\%$

Bank A compounded only once per year so the APY was exactly the starting interest rate. However, for both Bank B and Bank C, the APY was higher than the original interest rates. While the APY's are very close, Bank C offers a slightly more favorable interest rate to an investor.

Example 3

The APY for two banks are the same. What nominal interest rate would a monthly compounding bank need to offer to match another bank offering 4% compounding continuously?

Solve for APY for the bank where all information is given, the continuously compounding bank.

$FV = PV \cdot e^{rt} = 100 \cdot e^{0.04} \approx 104.08$

The APY is about 4.08%. Now you will set up an equation where you use the 104.08 you just calculated, but with the other banks interest rate.

$$FV = PV\left(1+\frac{i}{k}\right)^{kt}$$
$$104.08 = 100\left(1+\frac{i}{12}\right)^{12}$$
$$i = 12\left[\left(\frac{104.08}{100}\right)^{\frac{1}{12}} - 1\right] \approx 0.0400667$$

The second bank will need to offer slightly more than 4% to match the first bank.

Example 4

Which bank offers the best deal to someone wishing to deposit money?

- Bank A, offering 4.5% annually compounded
- Bank B, offering 4.4% compounded quarterly
- Bank C, offering 4.3% compounding continuously

The following table shows the APY calculations for the three banks.

TABLE 1.9:

Bank A	Bank B	Bank C

TABLE 1.9: (continued)

$FV = PV(1+i)^t$ $FV = 100(1+0.045)$ $APY = 4.5\%$	$FV = PV\left(1+\frac{i}{k}\right)^{kt}$ $FV = 100\left(1+\frac{0.044}{4}\right)^4$ $APY \approx 4.473\%$	$FV = PV \cdot e^{rt}$ $FV = 100e^{0.043}$ $APY \approx 4.394\%$

Bank B offers the best interest rate.

Example 5

What is the effective rate of a credit card interest charge of 34.99% APR compounded monthly?

$\left(1+\frac{.3499}{12}\right)^{12} \approx 1.4118$ or a 41.18% effective interest rate.

Review

For problems 1-4, find the APY for each of the following bank accounts.

1. Bank A, offering 3.5% annually compounded.
2. Bank B, offering 3.4% compounded quarterly.
3. Bank C, offering 3.3% compounded monthly.
4. Bank D, offering 3.3% compounding continuously.
5. What is the effective rate of a credit card interest charge of 21.99% APR compounded monthly?
6. What is the effective rate of a credit card interest charge of 16.89% APR compounded monthly?
7. What is the effective rate of a credit card interest charge of 18.49% APR compounded monthly?
8. The APY for two banks are the same. What nominal interest rate would a monthly compounding bank need to offer to match another bank offering 3% compounding continuously?
9. The APY for two banks are the same. What nominal interest rate would a quarterly compounding bank need to offer to match another bank offering 1.5% compounding continuously?
10. The APY for two banks are the same. What nominal interest rate would a daily compounding bank need to offer to match another bank offering 2% compounding monthly?
11. Explain the difference between APR and APY.
12. Give an example of a situation where the APY is higher than the APR. Explain why the APY is higher.
13. Give an example of a situation where the APY is the same as the APR. Explain why the APY is the same.
14. Give an example of a situation where you would be looking for the highest possible APY.
15. Give an example of a situation where you would be looking for the lowest possible APY.

Review (Answers)

To see the Review answers, open this PDF file and look for section 13.5.

1.6 Annuities

Learning Objectives

Here you'll learn how to compute future values of periodic payments.

Sally knows she can earn a nominal rate of 6% convertible monthly in a retirement account, and she decides she can afford to save $1,500 from her paycheck every month. How can you use geometric series to simplify the calculation of finding the future value of all these payments? How much money will Sally have saved in 30 years?

Annuity

An **annuity** is a series of equal payments that occur periodically. The word annuity comes from annual which means yearly. You will start by working with payments that occur once at the end of each year and then delve deeper to payments that occur monthly or any period.

Assume an investor saves R dollars at the end of each year for t years in an account that earns i interest per period.

- The first payment R will be in the bank account for $t-1$ years and grow to be: $R(1+i)^{t-1}$
- The second payment R will be in the bank account for $t-2$ years and grow to be: $R(1+i)^{t-2}$
- This pattern continues until the last payment of R that is deposited in the account right at t years, so it doesn't earn any interest at all.

The account balance at this point in the future (Future Value, FV) is the sum of each individual FV of all the payments:

$FV = R + R(1+i)^1 + R(1+i)^2 + \cdots + R(1+i)^{t-2} + R(1+i)^{t-1}$

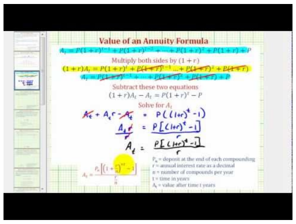
Recall that a geometric series with initial value a and common ratio r with n terms has sum:

$a + ar + ar^2 + \cdots + ar^{n-1} = a \cdot \frac{1-r^n}{1-r}$

So, a geometric series with starting value R and common ratio $(1+i)$ has sum:

$$\begin{aligned} FV &= R \cdot \frac{1-(1+i)^n}{1-(1+i)} \\ &= R \cdot \frac{1-(1+i)^n}{-i} \\ &= R \cdot \frac{(1+i)^n - 1}{i} \end{aligned}$$

This formula describes the relationship between FV (the account balance in the future), R (the annual payment), n (the number of years) and i (the interest per year).



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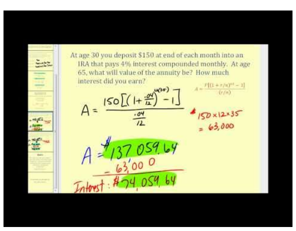
The formula is extraordinarily flexible and will work even when payments occur monthly instead of yearly by rethinking what, R, i and n mean. The resulting Future Value will still be correct. If R is monthly payments, then i is the interest rate per month and n is the number of months.

Take an IRA (special type of savings account). If Lenny saves \$5,000 a year at the end of each year for 35 years at an interest rate of 4%, he can determine what his Future Value will be using the formula.

$R = 5,000, \ i = 0.04, \ n = 35, \ FV = ?$

$$\begin{aligned}
FV &= R \cdot \frac{(1+i)^n - 1}{i} \\
FV &= 5,000 \cdot \frac{(1+0.04)^{35} - 1}{0.04} \\
FV &= \$368,281.12
\end{aligned}$$



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Examples

Example 1

Earlier, you were given a problem where Sally wanted to know how much she will have if she can earn a nominal 6% interest rate compounded monthly in a retirement account where she decides to save \$1500 from her paycheck every month for thirty years.

$FV = ?, \ i = \frac{0.06}{12} = 0.005, \ n = 30 \cdot 12 = 360, \ R = 1,500$

$$\begin{aligned}
FV &= R \cdot \frac{(1+i)^n - 1}{i} \\
FV &= 1,500 \cdot \frac{(1+0.005)^{360} - 1}{0.005} \\
FV &\approx 1,506,772.56
\end{aligned}$$

Example 2

How long does Mariah need to save if she wants to retire with a million dollars and saves $10,000 a year at 5% interest?

$FV = 1,000,000,\ R = 10,000,\ i = 0.05,\ n = ?$

$$\begin{aligned}
FV &= R \cdot \frac{(1+i)^n - 1}{i} \\
1,000,000 &= 10,000 \cdot \frac{(1+0.05)^n - 1}{0.05} \\
100 &= \frac{(1+0.05)^n - 1}{0.05} \\
5 &= (1+0.05)^n - 1 \\
6 &= (1+0.05)^n \\
n &= \frac{\ln\ 6}{\ln\ 1.05} \approx 36.7\ years
\end{aligned}$$

Example 3

How much will Peter need to save each month if he wants to buy an $8,000 car with cash in 5 years? He can earn a nominal interest rate of 12% compounded monthly.

In this situation you will do all calculations in months instead of years. An adjustment in the interest rate and the time is required and the answer needs to be clearly interpreted at the end.

$FV = 8,000,\ R = ?,\ i = \frac{0.12}{12} = 0.01,\ n = 5 \cdot 12 = 60$

$$\begin{aligned}
FV &= R \cdot \frac{(1+i)^n - 1}{i} \\
8,000 &= R \cdot \frac{(1+0.01)^{60} - 1}{0.01} \\
R &= \frac{8,000 \cdot 0.01}{(1+0.01)^{60} - 1} \approx 97.96
\end{aligned}$$

Peter will need to save about $97.96 every month.

Example 4

At the end of each quarter, Fermin makes a $200 deposit into a mutual fund. If his investment earns 8.1% interest compounded quarterly, what will his annuity be worth in 15 years?

Quarterly means 4 times per year.

$FV = ?,\ R = 200,\ i = \frac{0.081}{4},\ n = 60$

$FV = 200 \cdot \frac{\left(1+\frac{0.081}{4}\right)^{60} - 1}{\frac{0.081}{4}} \approx \$23,008.71$

Example 5

What interest rate compounded semi-annually is required to grow \$25 semi-annual payments to \$500 in 8 years?

$FV = 500,\ R = 25,\ i = ?,\ n = 8 \cdot 2 = 16$. Note that the calculation will be done in months. At the end you will convert your answer to years.

$$\begin{aligned} FV &= R \cdot \frac{(1+i)^n - 1}{i} \\ 500 &= 25 \cdot \frac{(1+i)^{16} - 1}{i} \\ 20i &= (1+i)^{16} - 1 \\ 0 &= (1+i)^{16} - 20i - 1 \end{aligned}$$

Using a graphing calculator, we find this equation has roots at $i = 0$ and $i = 0.0290$. Since $i \neq 0$, the semi-annual interest rate is $i = 0.0290 = 2.90\%$ for a nominal annual interest rate of 5.80%.

Review

1. At the end of each month, Rose makes a \$400 deposit into a mutual fund. If her investment earns 6.1% interest compounded monthly, what will her annuity be worth in 30 years?

2. What interest rate compounded quarterly is required to grow a \$40 quarterly payment to \$1000 in 5 years?

3. How many years will it take to save \$10,000 if Sal saves \$50 every month at a 2% monthly interest rate?

4. How much will Bob need to save each month if he wants to buy a \$33,000 car with cash in 5 years? He can earn a nominal interest rate of 12% compounded monthly.

5. What will the future value of his IRA be if Cal saves \$5,000 a year at the end of each year for 35 years at an interest rate of 8%?

6. How long does Kathy need to save if she wants to retire with four million dollars and saves \$10,000 a year at 8% interest?

7. What interest rate compounded monthly is required to grow a \$416 monthly payment to \$80,000 in 10 years?

8. Every six months, Shanice makes a \$1000 deposit into a mutual fund. If her investment earns 5% interest compounded semi-annually, what will her annuity be worth in 25 years?

9. How much will Jen need to save each month if she wants to put \$60,000 down on a house in 5 years? She can earn a nominal interest rate of 8% compounded monthly.

10. How long does Adrian need to save if she wants to retire with three million dollars and saves \$5,000 a year at 10% interest?

11. What will the future value of her IRA be if Vanessa saves \$3,000 a year at the end of each year for 40 years at an interest rate of 6.7%?

12. At the end of each quarter, Justin makes a \$1,500 deposit into a mutual fund. If his investment earns 4.5% interest compounded quarterly, what will her annuity be worth in 35 years?

13. What will the future value of his IRA be if Ted saves \$3,500 a year at the end of each year for 25 years at an interest rate of 5.8%?

14. What interest rate compounded monthly is required to grow a \$300 monthly payment to \$1,000,000 in 35 years?

15. How much will Katie need to save each month if she wants to put \$55,000 down in cash on a house in 2 years? She can earn a nominal interest rate of 6% compounded monthly.

Review (Answers)

To see the Review answers, open this PDF file and look for section 13.6.

1.7 Annuities for Loans

Learning Objectives

Here you'll learn how to compute present values of equal periodic payments.

Many people buy houses they cannot afford. This causes major problems for both the banks and the people who have their homes taken. In order to make wise choices when you buy a house, it is important to know how much you can afford to pay each period and calculate a maximum loan amount.

Joanna knows she can afford to pay $12,000 a year for a house loan. Interest rates are 4.2% annually and most house loans go for 30 years. What is the maximum loan she can afford? What will she end up paying after 30 years?

Annuities for Loans

The **present value** can be found from the future value using the regular compound growth formula:

$$PV(1+i)^n = FV$$
$$PV = \frac{FV}{(1+i)^n}$$

You also know the future value of an annuity:

$FV = R \cdot \frac{(1+i)^n - 1}{i}$

So by substitution, the formula for the present value of an annuity is:

$PV = R \cdot \frac{(1+i)^n - 1}{i} \cdot \frac{1}{(1+i)^n} = R \cdot \frac{(1+i)^n - 1}{i(1+i)^n} = R \cdot \frac{1-(1+i)^{-n}}{i}$

The present value of a series of equal payments R with interest rate i per period for n periods is:

$PV = R \cdot \frac{1-(1+i)^{-n}}{i}$

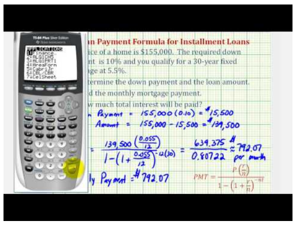
This formula can also be used to find out other information such as how much a regular payment should be and how long it will take to pay off a loan.

Take a $1,000,000 house loan over 30 years with a nominal interest rate of 6% compounded monthly. You are not given the monthly payments, R. To find R, solve for R in the formula given above.

$PV = \$1,000,000, \ R = ?, \ i = 0.005, \ n = 360$

$$PV = R \cdot \frac{1-(1+i)^{-n}}{i}$$
$$1,000,000 = R \cdot \frac{1-(1+0.005)^{-360}}{0.005}$$
$$R = \frac{1,000,000 \cdot 0.005}{1-(1+0.005)^{-360}} \approx 5995.51$$

It is remarkable that in order to pay off a $1,000,000 loan you will have to pay $5,995.51 a month, every month, for thirty years. After 30 years, you will have made 360 payments of $5995.51, and therefore will have paid the bank more than $2.1 million, more than twice the original loan amount. It is no wonder that people can get into trouble taking on more debt than they can afford.



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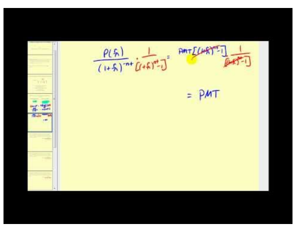

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Examples

Example 1

Earlier, you were asked about how much Joanna can afford to take out in a loan. Joanna knows she can afford to pay $12,000 a year to pay for a house loan. Interest rates are 4.2% annually and most house loans go for 30 years. What is the maximum loan she can afford? What does she end up paying after 30 years? You can use the present value formula to calculate the maximum loan:

$PV = 12,000 \cdot \frac{1-(1+0.042)^{-30}}{0.042} \approx \$202,556.98$

For 30 years she will pay $12,000 a year. At the end of the 30 years she will have paid $\$12,000 \cdot 30 = \$360,000$ total

Example 2

How long will it take to pay off a $20,000 car loan with a 6% annual interest rate compounded monthly if you pay it off in monthly installments of $500? What about if you tried to pay it off in monthly installments of $100?

$PV = \$20,000,\ R = \$500,\ i = \frac{0.06}{12} = 0.005,\ n = ?$

$$
\begin{aligned}
PV &= R \cdot \frac{1-(1+i)^{-n}}{i} \\
20,000 &= 500 \cdot \frac{1-(1+0.005)^{-n}}{0.005} \\
0.2 &= 1-(1+0.005)^{-n} \\
(1+0.005)^{-n} &= 0.8 \\
n &= -\frac{\ln 0.8}{\ln 1.005} \approx 44.74 \; months
\end{aligned}
$$

For the \$100 case, if you try to set up an equation and solve, there will be an error. This is because the interest on \$20,000 is exactly \$100 and so every month the payment will go to only paying off the interest. If someone tries to pay off less than \$100, then the debt will grow.

Example 3

It saves money to pay off debt faster in order to save money on interest. As shown earlier, interest can more than double the cost of a 30 year mortgage. This example shows how much money can be saved by paying off more than the minimum.

Suppose a \$300,000 loan has 6% interest convertible monthly with monthly payments over 30 years. What are the monthly payments? How much time and money would be saved if the monthly payments were larger by a factor of $\frac{13}{12}$? This is like making 13 payments a year instead of just 12. First you will calculate the monthly payments if 12 payments a year are made.

$$
\begin{aligned}
PV &= R \cdot \frac{1-(1+i)^{-n}}{i} \\
300,000 &= R \cdot \frac{1-(1+0.005)^{-360}}{0.005} \\
R &= \$1,798.65
\end{aligned}
$$

After 30 years, you will have paid \$647,514.57, more than twice the original loan amount.

If instead the monthly payment was $\frac{13}{12} \cdot 1798.65 = 1948.54$, you would pay off the loan faster. In order to find out how much faster, you will make your unknown.

$$
\begin{aligned}
PV &= R \cdot \frac{1-(1+i)^{-n}}{i} \\
300,000 &= 1948.54 \cdot \frac{1-(1+0.005)^{-n}}{0.005} \\
0.7698 &= 1-(1+0.005)^{-n} \\
(1+0.005)^{-n} &= 0.23019 \\
n &= -\frac{\ln 0.23019}{\ln 1.005} \approx 294.5 \; months
\end{aligned}
$$

294.5 months is about 24.5 years. Paying fractionally more each month saved more than 5 years of payments.

$294.5 \; months \cdot \$1,948.54 = \$573,847.99$

The loan ends up costing \$573,847.99, which saves you more than \$73,000 over the total cost if you had paid over 30 years.

Example 4

Mackenzie obtains a 15 year student loan for $160,000 with 6.8% interest. What will her yearly payments be?

$PV = \$160,000,\ R = ?,\ n = 15,\ i = 0.068$

$$160,000 = R \cdot \frac{1-(1+0.068)^{-15}}{0.068}$$
$$R \approx \$17,345.88$$

Example 5

How long will it take Francisco to pay off a $16,000 credit card bill with 19.9% APR if he pays $800 per month? Note: APR in this case means nominal rate convertible monthly.

$PV = \$16,000,\ R = \$600,\ n = ?,\ i = \frac{0.199}{12}$

$$16,000 = 600 \cdot \frac{1-\left(1+\frac{0.199}{12}\right)^{-n}}{\frac{0.199}{12}}$$
$$n = 24.50\ months$$

Review

For problems 1-10, find the missing value in each row using the present value for annuities formula.

TABLE 1.10:

Problem Number	*PV*	*R*	*n* (years)	*i* (annual)	Periods per year
1.		$4,000	7	1.5%	1
2.	$15,575		5	5%	4
3.	$4,500	$300		3%	12
4.		$1,000	12	2%	1
5.	$16,670		10	10%	4
6.		$400	4	2%	12
7.	$315,000	$1,800		5%	12
8.		$500	30	8%	12
9.		$1,000	40	6%	4
10.	$10,000		6	7%	12

11. Charese obtains a 15 year student loan for $200,000 with 6.8% interest. What will her yearly payments be?

12. How long will it take Tyler to pay off a $5,000 credit card bill with 21.9% APR if he pays $300 per month? *Note: APR in this case means nominal rate convertible monthly*.

13. What will the monthly payments be on a credit card debt of $5,000 with 24.99% APR if it is paid off over 3 years?

14. What is the monthly payment of a $300,000 house loan over 30 years with a nominal interest rate of 2% convertible monthly?

15. What is the monthly payment of a \$270,000 house loan over 30 years with a nominal interest rate of 3% convertible monthly?

Review (Answers)

To see the Review answers, open this PDF file and look for section 13.7.

The effects of interest on lump sum deposits and periodic deposits were explored. The key idea was that a dollar today is worth more than a dollar in a year.